AD-A096 120 STANFORD UNIV CA SYSTEMS OPTIMIZATION LAB F/G 12/1
TIME-STAGED LINEAR PROGRAMS.(U)
OCT 80 G B DANTZIG N00014-75-C-0267
UNCLASSIFIED SOL-80-28 NL

1 OF 1
AD
A096120

END
DATE
FILMED
4-81
DTIC

LEVEL II

Systems
Optimization
Laboratory

AD A096120

DTIC
MAR 1 1981

DDC FILE COPY

Department of Operations Research
Stanford University
Stanford, CA 94305

81 3 3 045

SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

TIME-STAGED LINEAR PROGRAMS

by

George B. Dantzig

TECHNICAL REPORT SOL 80-28

October 1980

Research and reproduction of this report were supported by the Department of Energy Contract DE-AC03-76SF00326, PA# DE-AT03-76ER72018, PA# DE-AT03-79EI10601, DE-AM03-76SF00326, PA# DE-AT03-80EI10682; Office of Naval Research Contract N00014-75-C-0267; National Science Foundation Grants MCS76-81259, MCS-7926009 and ECS-8012974 (formerly ENG77-06761).

TIME-STAGED LINEAR PROGRAMS

by

George B. Dantzig

In this paper we will study systems of the following form:

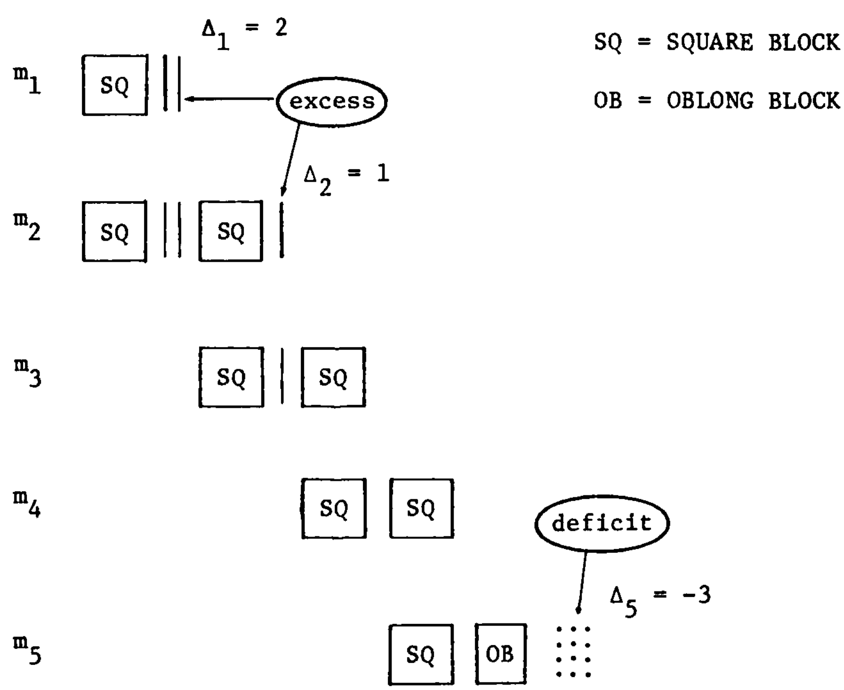
$$
\begin{aligned}
b_1 &= A_1 X_1 , && (X_1, X_2, X_3, X_4) \geq 0 \\
b_2 &= -B_1 X_1 + A_2 X_2 \\
b_3 &= - B_2 X_2 + A_3 X_3 \\
b_4 &= - B_3 X_3 + A_4 X_4 \\
b_5 &= - B_4 X_4 + A_5 X_5 \\
(\text{MIN}) Z &= c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_4 + c_5 X_5
\end{aligned}
$$

Of particular interest is the case where the A_k (and the B_k) are all equal. Our procedures do not require however, this to hold but a computational advantage can be achieved when it does.

Associated with each period i are m_i equations. We will use $m = m_i$ if all the m_i are equal.

Assuming no redundant equations, a basic solution to the primal system must have $m_1 + \Delta_1 \geq m_1$ basic variables among the components of X_1 in order to satisfy the m_1 constraints of the first period. For subsequent periods we have $m_i + \Delta_i$ basic variables where $\Delta_i > 0$ measures the <u>excess</u> over the number of equations and $\Delta_i < 0$ the <u>deficit</u>.* If $\Delta_i = 0$ for all periods, the basis is called square or balanced-block diagonal, [1], [2].

Structure of the Basis



*Assumes that the system has no redundant equations.

Properties

$$
\begin{array}{ll}
m_2 \geq \Delta_1 \geq 0 \ , & \theta \leq \Delta_1 \leq m_2 \\[2ex]
m_3 \geq \Delta_1 + \Delta_2 \geq 0 \ , & -m_2 \leq \Delta_2 \leq m_3 \\[2ex]
m_4 \geq \Delta_1 + \Delta_2 + \Delta_3 \geq 0 \ , & -m_3 \leq \Delta_3 \leq m_4 \\[2ex]
m_5 \geq \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 \geq 0 \ , & -m_4 \leq \Delta_4 \leq m_5 \\[2ex]
0 = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \Delta_5 = 0 \ , & -m_5 \leq \Delta_5 \leq 0
\end{array}
$$

Given a discrete time structure derived from a continuous time problem; suppose that an activity initiated at time $t = t_1$ is likely to persist to some time $t = t_2$ when it is replaced by some other activity that again persists for some time etc., -- then "for almost all" periods $\Delta_k = 0$.

Proof: We assume between time points t_1, t_2 that there are exactly $m + \Delta$ activities that operate at a positive level where m is the number of relations that must hold at each point t in time $t_1 \leq t \leq t_2$. Assume a fine time division for the discrete time approximation to the continuous time and that these same activities operate at positive levels between t_1 and t_2. Assume that the number of these basic activities in each period remains invariant:

$$m + \Delta , \qquad t_1 \leq t \leq t_2$$

The discrete time indices corresponding to t_1 and t_2 will be denoted by k_1 and k_2.

By the properties just discussed, we have at t_1 and t_2

$$m \geq \sum_{1}^{k_1} \Delta_i \geq 0$$

$$m \geq \sum_{1}^{k_1} \Delta_i + \sum_{k_1+1}^{k_2} \Delta_i \geq 0$$

whence

$$m \geq m - \sum_{1}^{k_1} \Delta_i \geq \sum_{k_1+1}^{k_2} \Delta_i \geq -\sum_{1}^{k_1} \Delta_i \geq -m$$

$$m \geq (k_2 - k_1)\Delta \geq -m , \qquad \Delta = \Delta_{k_1} = \cdots = \Delta_{k_2}$$

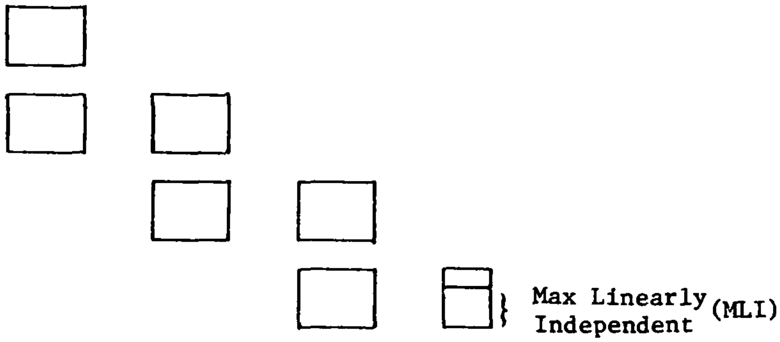
where

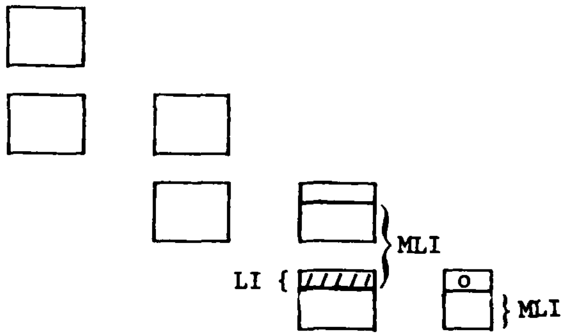
$$k_2 - k_1$$

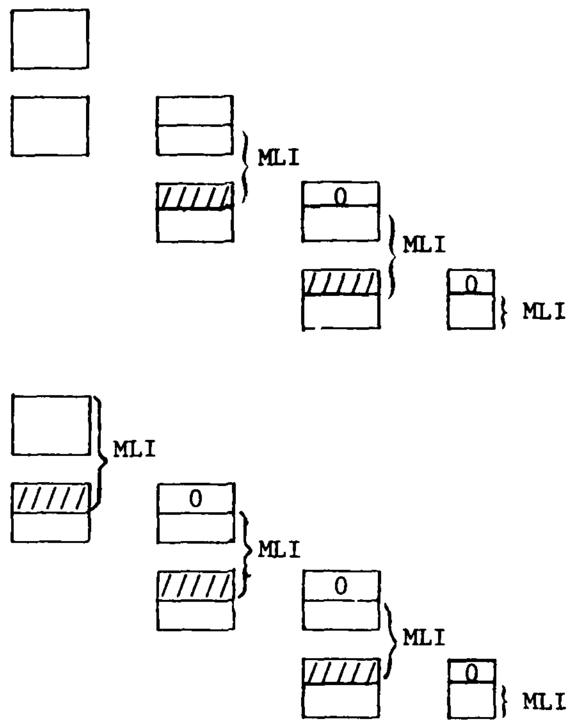
is the number of discrete time points between $t_1 < t \leq t_2$ and Δ, the excess or deficit in the number of activities over m, is a fixed integer independent of the number of subdivisions. It is clear that as the number of subdivisions $\rightarrow +\infty$, that the inequalities cannot hold unless $\Delta = 0$. The proof is the same as that given in [1].

If the basis B had square non-singular diagonal blocks, it would be easy to solve $BX = b$ or $\pi B = \gamma$. The effort would be proportionate to T. In general when the activities have the persistence quality, most diagonal blocks can be expected to be $m \times m$ but not necessarily non-singular.

How to Square a Basis





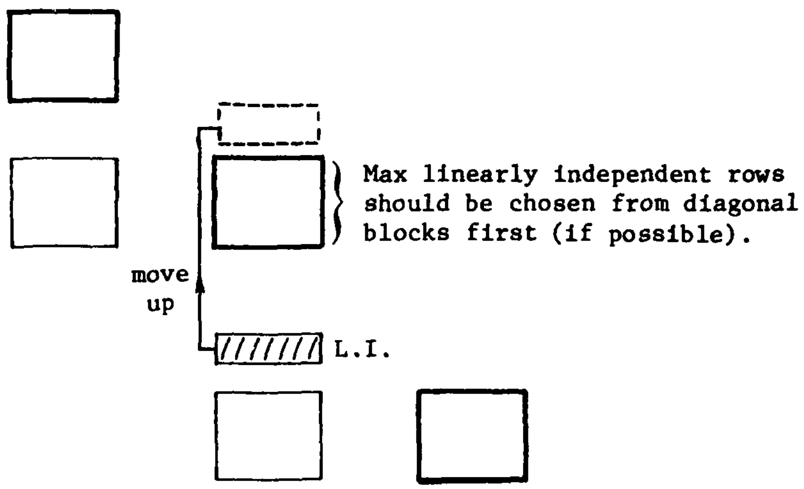


<u>Theorem</u>: The blocks of independent rows for each time period are square.

<u>Proof</u>: The number of columns associated with each set of linearly independent rows is greater or equal to the number of rows implying that the total number of columns $\geq$ the total number of linearly independent rows. However, these totals are equal implying equality for each set.

<u>Other Nice Properties</u>

If the largest independent number of rows used to do the eliminations are selected from the diagonal blocks (to the maximum extent possible), then advantage can be taken of any repetition of blocks along the diagonal.



For a more detailed discussion of the material discussed so far see R. Fourer [3].

FINDING A FEASIBLE SOLUTION TO A STAIRCASE PROBLEM

Myopic Approach

The approach is similar to that used in many econometric models and found in practice in industry.

Start with any guess about future prices. Say

$$\pi_2^o = 0, \qquad \pi_3^o = 0, \ldots, \pi_T^o = 0$$

Solve 1st period problem with cost row c_1 adjusted for future prices.

$$\text{Adj. Cost:} \quad d_1 = c_1 + \pi_2^o B_1$$

$$\boxed{\begin{array}{ll} b_1 = A_1 X_1 , & X_1 \geq 0 \\ \text{MIN}(d_1 X_1) & \end{array}} \qquad \text{yields} \quad X_1 = X_1^o$$

Unless not feasible, in which case the full system is not feasible and we stop.

Solve 2nd period problem with cost row c_2 adjusted for future prices.

$$\text{Adj. Cost:} \quad d_2 = c_2 + \pi_3^o B_2$$

$$\text{2nd Period Starting Inventory:} \quad a_2 = b_2 + B_1 X_1^o$$

$$\boxed{\begin{array}{ll} a_2 = A_2 X_2 , & X_2 \geq 0 \\ \text{MIN}(d_2 X_2) & \end{array}} \quad \text{yields} \quad X_2 = X_2^o$$

unless not feasible.

For $k = (2, \ldots, T)$, solve kth period problem:

$$\text{Adj. Cost:} \quad d_k = c_k + \pi_{k+1}^o B_k , \ k < T; \ d_T = c_T$$

$$\text{Inventory:} \quad a_k = b_k + B_{k-1} X_{k-1}^o$$

$$\boxed{\begin{array}{ll} a_k = A_k X_k , & X_k \geq 0 \\ \text{MIN}(d_k X_k) & \end{array}} \quad \text{yields} \quad X_k = X_k^o$$

unless k-th problem is not feasible.

Comment

1. If the original problem represents a real world problem, and the set of activities is sufficiently rich, then there will always be a feasible solution no matter what starting inventory vector a_k, at the start of period k, is inherited from the past.

2. It is anticipated for most practical models that the myopic approach will seldom run into an infeasibility. In other words finding a starting feasible solution

$$X_1^o, X_2^o, \ldots, X_T^o$$

will be a relatively cheap process.

3. If

$$a_k = A_k X_k, \qquad X_k \geq 0$$

is infeasible then instead of $X_k = X_k^o$, a vector $\sigma_k = \sigma_k^o$ will be produced such that $\sigma_k^o A_k \geq 0$, $\sigma_k^o a_k < 0$. Indeed unless there exists an a_k such that $\sigma_k^o a_k \geq 0$ the full system is infeasible. Therefore we seek an X_{k-1} such that

$$\sigma_k^o a_k = \sigma_k^o(b_k + B_{k-1} X_{k-1}) \geq 0 .$$

This relation, called a CUT, is added to the (k-1)st period problem.

CUT:

$$\boxed{(\sigma_k^o B_{k-1})X_{k-1} \geq -(\sigma_k^o b_k)}$$

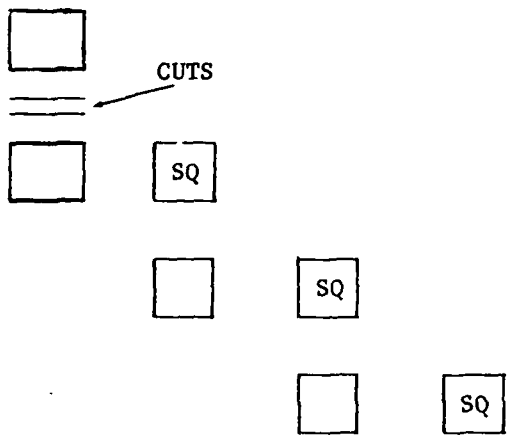
Either a new solution $X_{k-1} = X_{k-1}^1$ is produced and we can proceed forward again or the k - 1st period problem has been rendered infeasible by the cut causing, in turn, a cut for period k - 2 etc. If the k - 1 period problem is feasible with the cut, it may lead to a feasible solution to the k-period problem, or if the k-period problem remains infeasible, cause still another cut for the k-1 period to be generated:

CUT:

$$\boxed{(\sigma_k^1 B_{k-1})X_{k-1} \geq -(\sigma^1 b_k)}$$

Cuts when added will initially be tight but subsequently might go slack. When this happens it is recommended they be dropped. If cuts back-up all the way to the first period and the first period is rendered infeasible, it implies that the primal is infeasible. If so, the algorithm is terminated.

Feasible Basis



As noted, there are likely to be only a few such feasibility cuts, however.

Getting a Feasible Dual Solution

Given a primal feasible solution

$$X_1 = X_1^o, \qquad X_2 = X_2^o, \ldots, X_{T-1} = X_{T-1}^o, \qquad X_T = X_T^o \quad ,$$

find a dual feasible solution by an analogous myopic algorithm:

T-th Period Problem

Set $a_T = b_T + B_{T-1} X_{T-1}^o$. Solve

$$\boxed{\begin{array}{ll} a_T = A_T X_T \ , & X_T \geq 0 \\ \mathrm{MIN}(c_T X_T) & \end{array}} \quad \text{yields} \quad \pi_T = \pi_T^o$$

This problem has feasible solutions by hypothesis.

If it has an infinite minimum, then terminate with class of feasible solutions such that MIN $\to -\infty$. If finite minimum let $\pi_T = \pi_T^o$ be optimal.

<u>k-th Period Problem</u>

$$\text{Set} \quad a_k = b_k + B_{k-1} X_{k-1}^o \quad ,$$

$$d_k = c_k + \pi_{k+1}^o B_k$$

Solve

$$\boxed{\begin{array}{ll} a_k = A_k X_k \ , & X_k \geq 0 \\ \text{MIN}(d_k X_k) & \end{array}} \qquad \text{yields} \quad \pi_k = \pi_k^o$$

or if MIN $\to -\infty$, a homogeneous solution

$$Y_k = Y_k^o \geq 0$$

such that

$$A_k Y_k^o = 0$$

and

$$d_k Y_k^o < 0 \quad .$$

In practice we would expect the case of MIN $\to -\infty$ rarely if ever to occur. We will discuss below what to do if it does occur. We would expect the above algorithm to quickly solve to give a good estimate of $(\pi_2^o, \ldots, \pi_T^o)$. The new estimate of π^o could then be

used to initiate another estimate of X. Since exactly m_i basic X for each period i are obtained we know that this process can never converge to a case for which $\Delta_i \neq 0$. Hence, repeated recycling of the algorithm will not, in general, converge to an optimal solution. Before discussing a procedure that will, we will dispose of the case $\mathrm{MIN}(d_k X_k) \to -\infty$.

To illustrate, consider $k = T - 1$. We are interested in knowing whether or not the finding of a class of feasible solutions to the T-1st problem such that $\mathrm{MIN}(d_{T-1} X_{T-1}) \to -\infty$, implies that $Z_{T-1} \to -\infty$ for the two-stage problem below consisting of the last two periods:

$$a_{T-1} = A_{T-1} X_{T-1} \,, \qquad (X_{T-1}, X_T) \geq 0$$

$$b_T = -B_{T-1} X_{T-1} + A_T X_T$$

$$\mathrm{MIN}\ Z_{T-1} = c_{T-1} X_{T-1} + c_T X_T$$

If $Z_{T-1} \to -\infty$, it implies that $Z \to -\infty$ for the full system. The iterative loop about to be described, could generate several additional homogeneous solutions $Y^o_{T-1}, Y^1_{T-1}, \ldots, Y^r_{T-1}$ to the $T - 1$ period problem where r is finite. To see this note that

$$X_{T-1} = X^o_{T-1} + \sum_{i=0}^{r} Y^i_{T-1} \lambda_i \,, \quad \lambda_i \geq 0$$

for any particular choice of λ_i, is a feasible solution to the T-1st period problem. Therefore we can set

$$P^i = B_{T-1} Y^i_{T-1}$$

$$\gamma^i = c_{T-1} Y^i_{T-1}$$

and associate with such a solution a redefined T-th period problem

$$\pi_T^{r+1}: \qquad a_T = -\sum_0^r P^i \lambda_i + A_T X_T , \qquad \lambda_i \geq 0 , \qquad X_T \geq 0$$

$$\mathrm{MIN}(\sum_0^r \gamma^i \lambda_i + c_T X_T)$$

If this minimum is finite, the resulting prices $\pi_T = \pi_T^{r+1}$ will generate a $d_{T-1} = d_{T-1}^{r+1}$ and possibly generate Y_{T-1}^{r+1}, P^{r+1}, γ^{r+1}. Only a finite number of Y^i_{T-1} can be so generated because each corresponds to an extreme ray of the T-1st period problem. Therefore for a sufficiently large r, π_T^r prices out non-negative for every possible (γ^i, P^i) column that can be generated by extreme homogeneous solutions $Y^i_{T-1} \geq 0$. Therefore

$$0 \leq \gamma^i + \pi_T^r P^i = (c_{T-1} + \pi_T^r B_{T-1}) Y^i_{T-1} = d^r_{T-1} Y^i_{T-1}$$

i.e., $d^r_{T-1} Y^i_{T-1} \geq 0$ for all possible homogeneous solutions Y^i_{T-1}.

Thus, the case $\text{MIN}(d_{T-1} X_{T-1}) \to -\infty$, does not occur for π_T^r for some finite r sufficiently large.

On the other hand if a redefined T-th period problem has a $\text{MIN} \to -\infty$, we <u>terminate</u> with a class of feasible solutions to the original primal $\to -\infty$. This class can be written in the form

$$X = (X_1^o, \ldots, X_{T-1}^o + \theta \sum Y_{T-1}^i \lambda_i^o, X_T^o + \theta Y_T^o) \geq 0 \qquad \text{as} \quad \theta \to +\infty$$

where $(\lambda_1^o, \ldots, \lambda_r^o, Y_T^o) \geq 0$ is a homogeneous solution to the redefined T-th period problem with the property that $\gamma\lambda^o + c_T Y_T^o < 0$.

Assuming we have found a feasible primal and dual solution, we now turn to finding an optimal solution.

FINDING AN OPTIMAL FEASIBLE SOLUTION

The Imbedded Decomposition Approach of Manne and Ho [4] and of Glassey [5], will be applied to the dual rather than, as they did, to the primal. The principal advantages are, or are believed to be: (1) The primal solution is obtained without any need to reconstitute the underlying solution from various weighted combinations of variables generated by the process; (2) The solution should be more stable because starting off with a small error in X would only make the second period starting inventory vector slightly wrong. For practical dynamic planning problems the slightly wrong second period starting inventory vector is just as acceptable as the correct one. (3) The myopic type steps described earlier, will be all that is required (with some occasional back-tracking) to solve the system.

$$\begin{array}{lllll}
\pi_1 b_1 + \pi_2 b_2 + \pi_3 b_3 + \cdots + \pi_T b_T & = Z(\text{MAX}) & & & \\
\pi_1 A_1 - \pi_2 B_1 & \le c_1 & :X_1 & \text{MASTER (1)} & \\
\hline
\quad + \pi_2 A_2 - \pi_3 B_2 & \le c_2 & :X_2 & & \text{MASTER (2)} \\
\qquad + \pi_3 A_3 - \pi_4 B_3 & \le c_3 & & \text{SUB(1)} & \\
\qquad\qquad \ddots & \vdots & & & \\
\qquad\qquad\quad \pi_T B_{T-1} & \le c_{T-1} & & & \text{SUB(2)} \\
\qquad\qquad\quad \pi_T A_T & \le c_T & :X_T & &
\end{array}$$

MASTER(1) produces $X_1 = X_1^*$ and SUB(1) (which returns $\pi_2 = \pi_2^1$). SUB(1) is a problem of the same form except starting at $t = 2$ with b_2 replaced by $a_2 = b_2 + B_1 X_1^*$. Hence SUB_1 can be partitioned in the same way into a MASTER(2) and a SUB(2), etc.

We will, however, use an inductive approach solving the above problem assuming $\pi_{t+1} = \overset{o}{\pi}_{t+1}, \ldots, \pi_T = \overset{o}{\pi}_T$. This can be solved as a t-period problem with $\pi_t A_t - \pi_{t+1} B_t \leq c_t$ replaced by

$$\pi_t A_t \leq c_t + \overset{o}{\pi}_{t+1} B_t .$$

The MASTER(ℓ) Problem has the form: For $\ell = (1, \ldots, t)$

$$\pi_\ell \, a_\ell + \rho_\ell \, f_\ell + \sigma_\ell \, g_\ell = \mathrm{MAX}_\ell \qquad \underline{\text{Prices}}$$

$$(\rho_\ell, \sigma_\ell) \geq 0 \,, \qquad \pi_\ell \, A_\ell - \rho_\ell \, F_\ell - \sigma_\ell \, G_\ell \leq c_\ell \qquad :X^*_\ell$$

$$\rho_\ell \, e \qquad = 1 \qquad :\theta^*_\ell$$

where $e = (1, 1, \ldots, 1)'$.

MASTER(ℓ) is related to that of $\ell - 1$ by

$$a_\ell = b_\ell + B_{\ell-1} \, X^*_{\ell-1} \,, \qquad \ell = (2, \ldots, t)$$

(1)

$$a_1 = b_1$$

and related to that of $\ell + 1$ as follows: Let f^i_ℓ, F^i_ℓ denote the i-th component of f_ℓ and column of F_ℓ, then for $\ell = (1, \ldots, t-1)$,

$$f^i_\ell = \pi^i_{\ell+1} \, b_{\ell+1} + \rho^i_{\ell+1} \, f_{\ell+1} + \sigma^i_{\ell+1} \, g_{\ell+1}$$

(2)

$$F^i_\ell = \pi^i_{\ell+1} \, B_\ell$$

for some extreme solution $(\pi^{i}_{\ell+1}, \rho^{i}_{\ell+1}, \sigma^{i}_{\ell+1})$ of MASTER(ℓ + 1). In an analogous way, let g^{i}_{ℓ}, G^{i}_{ℓ} denote the i-th component of g_{ℓ} and column of G_{ℓ}. Then

$$g^{i}_{\ell} = \pi^{i}_{\ell+1}\, b_{\ell+1} + \sigma^{i}_{\ell+1}\, g_{\ell+1}$$

(2.1)

$$G^{i}_{\ell} = \pi^{i}_{\ell+1}\, B_{\ell}$$

for some homogeneous solution $\pi^{i}_{\ell+1}, \sigma^{i}_{\ell+1} \geq 0$ such that $\pi^{i}_{\ell+1} A - \sigma^{i}_{\ell+1} G_{\ell+1} \leq 0$, and $\pi^{i}_{\ell+1} a_{\ell+1} + \sigma^{i}_{\ell+1} g_{\ell+1} > 0$.*

For the case $\ell = t$, we define,

$$f_{t} = \sum_{t+1}^{T} \pi^{o}_{i}\, b_{i}$$

(3)

$$F_{t} = \pi^{o}_{t+1}\, B_{t+1}$$

*Note in (2.1) terms involving $\rho^{i}_{\ell+1} f_{\ell+1}$ have been omitted because $e\rho^{i}_{\ell+1} = 1$ means that $\rho^{i}_{\ell+1} \geq 0$ cannot enter as part of a homogeneous solution

Iterative algorithm is initiated by setting $t = 1$, $\ell = 1$, applying formula (3), and then going to CYCLE.

CYCLE:

I. Optimize MASTER(ℓ);

Determine the optimal solution: MAX_ℓ, π^r_ℓ, ρ^r_ℓ, σ^r_ℓ, where $r - 1$ is the number of columns in $F_{\ell-1}$ or if $MAX_\ell \to +\infty$, determine homogeneous solution π^r_ℓ, ρ^r_ℓ per (2.1).

II. If $\ell > 1$ and $MAX_\ell > \theta_{\ell-1}$ go BACKWARD to MASTER($\ell - 1$) as follows:

Augment MASTER($\ell - 1$) by tacking on an r-th component to $\rho_{\ell-1}$ by using $(\pi_\ell, \rho_\ell, \sigma_\ell) = \pi^r_\ell, \rho^r_\ell, \sigma^r_\ell$ to determine $f^r_{\ell-1}$, $F^r_{\ell-1}$ by (2) or if $MAX_\ell \to +\infty$, determine g^i_ℓ, G^i_ℓ, by (2.1). Set $\ell = \ell - 1$;

GO TO CYCLE;

III. If $\ell = 1$ or if $\ell > 1$ and $MAX_\ell \leq \theta_{\ell-1}$, then

If $\ell = T$, TERMINATE $X = X^*$ is optimal;

If $\ell < T$, go FORWARD to MASTER($\ell + 1$) as follows:

If $\ell = t$, set $t = t + 1$ and determine f_t, F_t by (3);

Set $\ell = \ell + 1$; use $X^*_{\ell-1}$ to determine a_ℓ by (1);

GO TO CYCLE.

Comments

In practice it is more convenient to work with the dual of MASTER(ℓ). The inequalities corresponding to the components of ρ_ℓ are cuts representing surrogate constraints for future prices. There may be feasibility cuts as well. We therefore highlight the iterative algorithm described above in its primal setting:

ℓ-th PERIOD PROBLEM

$$\begin{array}{l|ll|l}
\pi_\ell: & & A_\ell X_\ell = a_\ell, \quad X_\ell \geq 0 & \\
\rho_\ell: & \text{CUTS}\left\{\begin{array}{l} \\ \\ \end{array}\right. & e\,\theta_\ell - F_\ell X_\ell \geq f_\ell \quad \leftarrow\cdots & \cdots \text{Surrogate Cuts for future prices and needs} \\
\sigma_\ell: & & - G_\ell X_\ell \geq g_\ell \quad \leftarrow\cdots & \cdots\text{Feasibility Cuts} \\
 & & \theta_\ell + c_\ell X_\ell = \text{MIN}_\ell &
\end{array}$$

If ℓ-th period problem is infeasible, it will give rise at end of Phase I to Π^*_ℓ, σ^*_ℓ and a relation which the current value of a_ℓ does not satisfy. For feasibility, the current a_ℓ should satisfy (but does not)

$$\pi^*_\ell a_\ell + \sigma^*_\ell g_\ell \leq 0$$

where

$$\sigma_\ell^* \geq 0 \ , \qquad \pi_\ell^* A_\ell - \sigma_\ell^* G_\ell \leq 0$$

We therefore seek an $X_{\ell-1}$ whose a_ℓ does. Substituting

$$a_\ell = b_\ell + B_{\ell-1} X_{\ell-1}$$

we generate a cut for dual of MASTER(ℓ - 1), see (2.1), namely

$$\boxed{-(\pi_\ell^* B_{\ell-1})X_{\ell-1} \geq +\pi_\ell^* b_\ell + \sigma_\ell^* g_\ell} \qquad \text{Feasibility Cut}$$

and return to MASTER(ℓ - 1).

If ℓ-th period problem is feasible, let the optimal solution be $(X^*, \theta_\ell^*; \pi_\ell^r, \rho_\ell^r, \sigma_\ell^r)$. If any of the CUTS for this solution are slack it is recommended that they be dropped.

The test for moving forward to $\ell + 1$ is feasibility and

$$\boxed{MIN_\ell \leq \theta_{\ell-1}^*}$$

If feasible and test is yes, one moves forward to $\ell + 1$ by setting

$$\boxed{a_{\ell+1} = b_{\ell+1} + B_\ell X_\ell^*} \qquad \text{Adj. RHS for } \ell + 1$$

If feasible and test is no, one returns to $\ell - 1$ with a new cut, see (2), namely

$$\boxed{\theta_{\ell-1} - (\pi_\ell^r B_{\ell-1})X_{\ell-1} \geq \pi_\ell^r b_\ell + \rho_\ell^r f_\ell + \sigma_\ell^r w_\ell} \quad \text{Look-ahead Cut}$$

which is a surrogate cut reflecting future prices and requirements. If infeasible then one returns to $(\ell-1)$ with a feasibility cut described earlier.

Note that once a cut is imposed we are recommending it be successively tested and "passed back" to period $\ell - 2, \ell - 3, \ldots$ until the test says to move forward again and then to move forward removing any cuts that are made slack by the current primal solution $X_1^*, X_2^*, \ldots$. It may be better tactics, however, after reoptimizing $\ell - 1$ to return to ℓ seeking more cuts until no more are passed back to $\ell - 1$. At this point check $MIN_{\ell-1} - \theta_{\ell-2}^* > 0$. If yes, return to $\ell - 2$ with cuts else move forward to ℓ.

This paper is based on a draft prepared around 1977 describing the imbedded dual decomposition approach for solving staircase systems. Philip Abrahamson [6] and Robert Wittrock in their research, have experimented with a variety of factors, variants and hybrids with other approaches. Their results are expected to be published soon.

REFERENCES

[1] A.F. Perold and G.B. Dantzig, "A Basis Factorization Method for Block Triangular Linear Program," in Proceedings of Symposium on Sparse Matrix Computation, G.W. Stewart and I.S. Duff (eds.), 1979.

[2] Fourer, R., "Sparse Gaussian Elimination of Staircase Systems," Technical Report SOL 79-17, Systems Optimization Laboratory, Department of Operations Research, Stanford University (1979).

[3] Fourer, R., "Solving Staircase Linear Programs by the Simplex Method, 1: Inversion," Technical Report SOL 79-18; "2: Pricing," Technical Report SOL 79-19, Systems Optimization Laboratory, Department of Operations Research, Stanford University (1979).

[4] Ho, J.K., and A.S. Manne, "Nested Decomposition for Dynamic Models," Mathematical Programming 6 (1974), 121-140.

[5] Glassey, C.R., "Nested Decomposition and Multi-stage Linear Programs," Management Science 20 (1973), 282-292.

[6] Abrahamson, Philip G., "A Nested-Decomposition Approach for Solving Staircase-Structured Linear Programs," Technical Report SOL 80-20, Systems Optimization Laboratory, Department of Operations Research, Stanford University (1980).

GENERAL REFERENCES

Dantzig, George B., "Upper Bounds, Secondary Constraints, and Block Triangularity in Linear Programming (Notes on Linear Programming: Part VIII, X)," RM-1367, The RAND Corporation (October 1954).

__________, "Optimal Solution of a Dynamic Leontief Model with Substitution (Notes on Linear Programming: Part XIII)," RM-1281-1, The RAND Corporation (April 1955).

__________, On the Status of Multi-Stage Linear Programs," R-1028, The RAND Corporation (February 1957),

__________, Compact Basis Triangularization for the Simplex Method," RR-28, Operations Research Center, University of California, Berkeley, (January 1962).

__________, "Linear Control Processes and Mathematical Programming," 64-13, Operations Research Center, University of California, Berkeley, (December 1974).

Fourer, R., "Solving Staircase Linear Programs by the Simplex Method, 2: Pricing," Technical Report SOL 79-19, Systems Optimization Laboratory Department of Operations Research, Stanford University (1979).

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER 14 SOL-80-28	2. GOVT ACCESSION NO. AD-A096	3. RECIPIENT'S CATALOG NUMBER 120
4. TITLE (and Subtitle) 6 TIME-STAGED LINEAR PROGRAMS.		5. TYPE OF REPORT & PERIOD COVERED 9 Technical Report,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 George B. DANTZIG		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-75-C-0267 DE-AC03-76SF00326
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Operations Research - SOL Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-047-143
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research, Dept. of the Navy 800 N. Quincy Street Arlington, VA 22217	11	12. REPORT DATE October 1980
		13. NUMBER OF PAGES 25 12 28
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale; its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

LINEAR PROGRAMS	DECOMPOSITION PRINCIPLE
DYNAMIC	COMPACT BASES
LARGE-SCALE	FEASIBILITY CUTS
STAIRCASE	LOOK-AHEAD CUTS

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper outlines some procuedures for solving time-staged (staircase) linear programs. Two approaches are discussed: the first based on modifying the block structure of the basis so that there are square non-singular sub-blocks along the diagonal; and the second based on the nested decomposition principle except applied to the dual system instead of the primal as proposed by Glassey and by Manne and Ho.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE 408765 JL

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)


8